MILNOR NUMBERS FOR SURFACE SINGULARITIES

BY

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ABSTRACT

An additive formula for the Milnor number of an isolated complex hypersurface singularity is shown. We apply this formula for studying surface singularities. Durfee's conjecture is proved for any absolutely isolated surface and a generalization of Yomdin singularities is given.

1. Introduction

Let $f\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be the germ of a holomorphic function with an isolated critical point, $(V, 0)$ (the germ of) its zero locus and $F_{f,0}$ its Milnor fibre, [15]. It is known that $F_{f,0}$ has the homotopy type of a wedge of $(n-1)$-spheres; the number $\mu(V, 0)$ of $(n-1)$-spheres is called the Milnor number of $(V, 0)$. The topology of $F_{f,0}$ and its relation with geometric invariants of $(V, 0)$ have been studied extensively.

The aim of this work is twofold. First, we study the relationship between the topology of $F_{f,0}$ and a partial resolution of $(V, 0)$. We show an additive formula for $\mu(V, 0)$ and as a consequence we prove Durfee's conjecture $6\, p_g \le \mu$ for any absolutely isolated singularity of surface. Secondly, we introduce a new class of singularities called t-singularities, see definition in Section 3. For such a t-singularity the above (geometric and topological) invariants can be obtained from the first two terms of the Taylor expansion of f around the singular point.

For an isolated plane curve singularity $(C, 0) \subset (\mathbb{C}^2, 0)$ we have the well-known formula:

$$(\mathrm{N}) \qquad \mu(C, 0) = d(d - 1) + \sum_{x \in \mathrm{Sing}(\widetilde{C})} \mu(\widetilde{C}, x) + 1 - r,$$

which relates the number r of different tangent lines of $(C, 0)$, the multiplicity d of C at 0, the Milnor number $\mu(C, 0)$ and the Milnor numbers of the strict transform $\widetilde{C}$ of C after one-point blow-up at its singular point.

In higher dimensions, after one-point blow-up, the strict transform $\widetilde{V}$ of $(V, 0)$ can have non-isolated singularities. Parusiński, [17], defined the generalized Milnor number $\mu(M; Z)$ for hypersurfaces Z in a compact complex manifold M. This invariant generalizes the notion of Milnor number and behaves well under blow-up. Using the generalized Milnor number, a similar formula to (N) for an isolated hypersurface singularity $(V, 0)$, relating its multiplicity, its Milnor number and some generalized Milnor number of its tangent cone, can be obtained. This formula appears in Section 2, Theorem 1.

In the surface case the Milnor number $\mu(V, 0)$ is equal to the dimension of the finite dimensional complex vector space $\frac{\mathbb{C}\{x, y, z\}}{J(f)}$ where $J(f)$ is the jacobian ideal of f in $\mathbb{C}\{x, y, z\}$, [16]. Also $\mu(V, 0)$ can be computed from the Newton polyhedron of f when the polyhedron is non-degenerated [8]. Moreover, an interesting formula, given by Laufer [9], gives $\mu(V, 0)$ in terms of some invariants of a resolution of the singularity of $(V, 0)$. One of these invariants is the geometric genus p_g of the singularity. Durfee, [3], conjectured that the geometric genus and the Milnor number satisfy the inequality $6\, p_g \leq \mu$. This is an old open problem in Singularity Theory. In Section 3 the formula in Theorem 1 is used for proving this conjecture for any surface singularity such that it can be resolved only by blowing-ups with a point as center.

In sections 4 and 5 we obtain some formulae for $\mu(V, 0)$ in terms of geometric and topological invariants of the projective plane curves defined by the homogeneous polynomials appearing in the Taylor expansion of f around 0. Let

$$f = f_d + f_{d+k} + \cdots$$

be such an expansion. Let D and T denote the divisors in $\mathbb{P}^2$ defined by the homogeneous polynomials f_d and f_{d+k} and $\mathrm{Sing}(D)$ the singular locus of D. Note that D is the projectivized tangent cone of $(V, 0)$ at the origin. A singularity $(V, 0)$ which satisfies

$$(*) \qquad\qquad \mathrm{Sing}(D) \cap T = \emptyset$$

is called a **Yomdin singularity**. The study of Yomdin singularities comes from the papers of Yomdin [26] and Lê [11] about hypersurfaces with one-dimensional singular locus. Several (topological) invariants have been calculated for these singularities: complex monodromy, [1], [20], polar invariants [14], spectrum of the

singularity [19], zeta-function of the monodromy [5]. The topological determinacy order for a Yomdin singularity $(V, 0)$ is equal to $(d + k)$ and its Milnor number is given by the formula, see e.g. [14],

$$\mu(V, 0) = (d - 1)^3 + k \sum_{P \in \text{Sing}(D)} \mu(D, P),$$

where $\mu(D, P)$ is the Milnor number of D at P (condition $(*)$ implies that D is a reduced plane curve and then D has only isolated singularities). The geometry of the pair $(\mathbb{P}^2, D)$ contains interesting information about the topology and geometry (resolution) of the germ $(V, 0)$. This fact has been used to disprove several conjectures. Luengo [13] showed examples, with $k = 1$, for which the μ-stratum is not smooth, Artal-Bartolo [1] found a counterexample to a Yau's conjecture [25] proving the non-determinacy of the topological type of the singularity by the link of the singularity and by the characteristic polynomial of the complex monodromy.

In order to study further the relationship between the geometry and topology of this kind of singularities it is necessary to deal with singularities whose projectivized tangent cone $D \subset \mathbb{P}^2$ is not reduced. This is one of the goals of this work. Let D_{red} be the reduced divisor in $\mathbb{P}^2$ determined by D and p its degree. We introduce the notion of t-singularity and prove:

THEOREM 4: *Every t-singularity $(V, 0)$ is isolated. Its Milnor number is equal to*

$$\mu(V, 0) = (d - 1)^3 + k \cdot \left(e(D) - 3d + d^2\right) + k(d - p)(d + k),$$

where $e(D)$ is the Euler characteristic of D. Moreover, the topological type of $(V, 0)$ is determined by the $(d + k)$-jet of f.

This result is interesting because it gives the topological determinacy of $(V, 0)$ under relatively weak transversality conditions (D may be non-reduced). In fact, these seem the most general conditions to obtain $(d + k)$ topological determinacy order.

It seems of interest to deal with polynomials $f = f_d + f_{d+k}$ of three complex variables which have only two non-zero homogeneous terms. Let $(V, 0)$ be its zero locus. Next we study the most general transverality condition which implies that $(V, 0)$ has an isolated singularity. Theorem 5 gives the Milnor number of such a surface.

THEOREM 5: *Let $f = f_d + f_{d+k}$ be a polynomial in three complex variables. Let $(V, 0)$ be the germ of surface at 0 defined by the zero locus of f. Then $(V, 0)$ has*

an isolated singularity if and only if $\mathrm{Sing}(D) \cap \mathrm{Sing}(T)$ *is empty. In this case the Milnor number of* $(V, 0)$ *is given by the formula*

$$\mu(V, 0) = (d-1)^3 + k\Big(e(D) + d^2 - 3d + \sum_{P \in \mathrm{Sing}(D) \cap T} ((D, T)_P - 1)\Big),$$

where $(D, T)_P$ *denotes the intersection multiplicity of both curves at* P.

Therefore the above formula for the Milnor number generalizes the one in Theorem 4. Nevertheless, in general the topological determinacy order for the polynomial $f_d + f_{d+k}$ will be greater than $(d+k)$. We show some examples related with this fact.

In [5] we have studied the zeta-function of the complex monodromy of $(V, 0)$. We gave an explicit procedure for computing the zeta-function of any singularity defined by a polynomial $f_d + f_{d+k}$, isolated or not. Nevertheless, this procedure does not give explicit formulae for the Milnor number as in Theorems 4 and 5.

Finally in the last section of the paper we study the global situation. Namely, for a projective surface Z in $\mathbb{P}^3$ given by $f_d w^k + f_{d+k} = 0$ we describe all its singularities and compute their Milnor numbers, see Theorem 6. Milnor numbers are computed using our previous results. Note that in case $k = 1$ this kind of description was used by Soares and Giblin for studying series of surfaces, [21].

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2. Additive formula for the Milnor number

Let $f: U \subset \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function, U a neighbourhood around the origin, $f(0) = 0$ and $(V, 0)$ the germ of the zero locus of f. It is known, see [10], [15], that there exist ε, sufficiently small, and δ with $0 < \delta \ll \varepsilon$ such that if B_ε is the open ball in $\mathbb{C}^n$ of radius ε centered at the origin and $\mathbb{D}_\delta^*$ is the open punctured disc in $\mathbb{C}$ of radius δ, then f restricted to $B_\varepsilon \cap f^{-1}(\mathbb{D}_\delta^*)$ is a smooth locally trivial fibration. A fibre of this fibration is called the **Milnor fibre of** f **attached to** 0 and it is denoted by $F_{f,0}$. The **topological Milnor number** of $(V, 0)$ is the integer

$$\mu(V, 0) := (-1)^{n-1}(e(F_{f,0}) - 1)$$

where $e(A)$ denotes the Euler characteristic of the set A. Obviously, if $(V, 0)$ has an isolated singularity then the topological Milnor number is the usual Milnor number [15]. The next two examples will be used in the paper.

Example 1: Let g: $(\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, h: $(\mathbb{C}^m, 0) \to (\mathbb{C}, 0)$ and $f(u, v) = g(u) + h(v)$: $(\mathbb{C}^n \times \mathbb{C}^m, 0) \to (\mathbb{C}, 0)$. Let $(V, 0)$, $(D, 0)$ and $(T, 0)$ be the germs of hypersurfaces defined by $f = 0$, $g = 0$ and $h = 0$ respectively. Since $F_{f,0}$ is the join space of $F_{g,0}$ and $F_{h,0}$, e.g. see [2], the topological Milnor number of $(V, 0)$ is equal to the product of the topological Milnor numbers of $(D, 0)$ and $(T, 0)$.

Example 2: Let $h, g \in \mathbb{C}\{x, y\}$ such that $h = 0$, $(g = 0)$ defines the germ of a singular (respectively smooth) plane curve $(D, 0)$ (respectively $(T, 0)$). Suppose that $(D, 0)$ and $(T, 0)$ have no common branches. Let $\varphi := h + z^k g$ and let $(V, 0)$ be the germ of surface defined by $\varphi = 0$. Let s be the intersection multiplicity of $(D, 0)$ and $(T, 0)$ at the origin. After an analytic change of coordinates we may assume that $g = x$, $h = y^s + x\, r(x, y)$ and $\varphi = y^s + x\, r(x, y) + z^k x$ with $r(0, 0) = 0$ because the origin is a singular point of $(D, 0)$. Hence the topological Milnor number of $(V, 0)$ satisfies

$$\mu(V, 0) = (k - 1)\,\mu(D, 0) + k\,(s - 1).$$

We prove it as follows. For $k = 1$, we make another change of coordinates such that $y^s + xz = 0$ is an equation for $(V, 0)$. By Example 1, $\mu(V, 0) = s - 1$.

Suppose that $k > 1$. Let $\psi = v^s + u\, r(u, v) + wu \in \mathbb{C}\{u, v, w\}$. Let $F_{\varphi,0}$ and $F_{\psi,0}$ be the Milnor fibres of φ and ψ, that means:

$$F_{\varphi,0} = \{(x, y, z) \in B_\varepsilon\colon xz^k + x\, r(x, y) + y^s = \delta\,\},$$
$$F_{\psi,0} = \{(u, v, w) \in B_\varepsilon\colon uw + u\, r(u, v) + v^s = \delta\,\}.$$

The map π: $F_{\varphi,0} \to F_{\psi,0}$, $\pi(x, y, z) = (x, y, z^k)$ is a k-sheeted cyclic covering. The ramification locus of π is the Milnor fibre $F_{h,0}$. Then

$$e(F_{\varphi,0}) = k\, e(F_{\psi,0}) - (k - 1)\, e(F_{h,0}).$$

After the analytic change of coordinates $\bar{w} = w + r(u, v)$ the function ψ is analytically equivalent to $u\bar{w} + v^s$. Thus Example 1 gives $e(F_{\psi,0}) = s$ and the equality for the topological Milnor number $\mu(V, 0)$ has been proved. Note that the singularity of $(V, 0)$ is isolated if and only if the singularity of $(D, 0)$ is isolated.

GENERALIZED MILNOR NUMBER. Let M be a compact n-dimensional complex manifold and let $\mathcal{L}$ be a holomorphic line bundle over M. Let Z be the zero locus of a holomorphic global section s of $\mathcal{L}$. Parusiński [17] defined the **generalized Milnor number** of Z in M as follows:

$$\mu(M; Z) := (-1)^n (e(Z) - \int_M c(\mathcal{L})^{-1} c_1(\mathcal{L}) c(M)),$$

where $c(\mathcal{L})$ and $c_1(\mathcal{L})$ denote the total Chern class and the first Chern class of $\mathcal{L}$. If Z is smooth the Euler characteristic of Z is given by, see e.g. [7],

$$e(Z) = \int_M c(\mathcal{L})^{-1} c_1(\mathcal{L}) c(M)$$

and then $\mu(M; Z) = 0$. Moreover, if Z' is the zero locus of another holomorphic global section s' of $\mathcal{L}$ then $\mu(M; Z) - \mu(M; Z') = (-1)^n (e(Z) - e(Z'))$.

Parusiński and Pragacz, [18], showed how to compute this invariant. Let $\mathcal{S}$ be a Whitney stratification of Z. Let $x \in Z$; locally the hypersurface Z is given at x by the zero locus of a holomorphic function $h \colon U \subset \mathbb{C}^n \to \mathbb{C}, h(0) = 0$. Let F_x be its Milnor fibre. Since Thom's Second Isotopy Lemma (see e.g. [4] Theorem 5.8) the topological type of the Milnor fibre F_x is constant along each stratum of $\mathcal{S}$. Let $\mu_S(Z)$ be the constant value of $x \mapsto \mu(Z, x)$ on S. Let s' be another holomorphic global section of $\mathcal{L}$ such that the zero locus Z' of s' is smooth and transverse to a Whitney stratification $\mathcal{S}$ of Z. Then

$$\mu(M; Z) = \sum_{S \in \mathcal{S}} \mu_S(Z) \cdot e(S \smallsetminus Z').$$

Indeed Parusiński, [17], defined $\mu(M; Z, D)$ for every compact subvariety D of Z which admits a neighbourhood U in Z such that $U \smallsetminus D$ is nonsingular. If the Whitney stratification $\mathcal{S}$ of Z induces a Whitney stratification $\mathcal{S}_D$ of D then

$$(1) \qquad \mu(M; Z, D) = \sum_{S \in \mathcal{S}_D} \mu_S(Z) \cdot e(S \smallsetminus Z').$$

Let $\mathrm{Sing}(Z)$ be the set of singular points of Z. If $D_1, \ldots, D_r$ are all compact and connected components of $\mathrm{Sing}(Z)$ then $\mu(M; Z) = \sum_{i=1}^r \mu(M; Z, D_i)$; for example, if Z has only isolated singularities, then $\mu(M; Z) = \sum_{x \in \mathrm{Sing}(Z)} \mu(Z, x)$.

Example 3: If $M = \mathbb{P}^2$ and Z is a curve of degree d then $\mu(\mathbb{P}^2; Z) = e(Z) - 3d + d^2 \geq 0$. The inequality is clear if Z is reduced. If Z is not a reduced curve then let p be the degree of the reduced curve Z_{red}. Using the above properties of the generalized Milnor number we get

$$\mu(\mathbb{P}^2; Z) = e(Z_{red}) - 3d + d^2 = \mu(\mathbb{P}^2; Z_{red}) + 3p - p^2 - 3d + d^2 \geq 0.$$

ADDITIVE FORMULA FOR THE MILNOR NUMBER.   Let $(V, 0) \subset (\mathbb{C}^n, 0)$ be an isolated hypersurface singularity defined by the zero locus of a holomorphic function f. For studying several topological invariants related to $(V, 0)$ we may assume

that $f = f_d + f_{d+1} + \cdots + f_r$ is a polynomial of degree r large enough. Let $D \subset \mathbb{P}^{n-1}$ be the projectivized tangent cone of V at 0, i.e. D is defined by f_d. Let $Z \subset \mathbb{P}^n$ be the projective hypersurface defined by the homogenized f with respect to a new variable and let Q be the point of Z corresponding to 0 in V. Let $\pi\colon \mathcal{X} \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ with centre at Q. From now on $\widetilde{Z}$ denotes the strict transform of Z, E denotes the exceptional divisor of π and we make the usual identification $\widetilde{Z} \cap E = D$.

In this situation one has the following Noether type formula which relates the Milnor number $\mu(V, 0)$ with the multiplicity of $(V, 0)$ at 0 and another two generalized Milnor numbers of the projectivized tangent cone of $(V, 0)$ at 0.

THEOREM 1: *In the previous conditions the following equality holds:*

$$\mu(V, 0) = (d - 1)^n + \mu(\mathbb{P}^{n-1}; D) + \mu(\mathcal{X}; \widetilde{Z}, D).$$

Proof: We compare the Euler characteristics of Z and $\widetilde{Z}$. Consider the restriction map $\pi|_{\pi^{-1}(Z)}\colon \pi^{-1}(Z) \to Z$ and the sets $S_0 := Z \smallsetminus \{Q\}$ and $S_1 := \{Q\}$. The morphism $\pi|_{\pi^{-1}(S_0)}\colon \pi^{-1}(S_0) \to S_0$ is a biholomorphism and then it is a one-to-one fibration. Moreover, the restriction map $\pi|_{\pi^{-1}(Q)}\colon E \to S_1$ is a fibration over a point. Following [12] the restriction map $\pi|_{\pi^{-1}(Z)}$ is a **descriptible morphism** and then

$$e(\pi^{-1}(Z)) = e(E) \cdot e(Q) + e(Z - Q) = e(E) + e(Z) - 1.$$

Since $e(\pi^{-1}(Z)) = e(\widetilde{Z}) + e(E) - e(\widetilde{Z} \cap E)$ and $\widetilde{Z} \cap E = D$ we get $e(\widetilde{Z}) - e(Z) = e(D) - 1$.

From the nice behaviour of the generalized Milnor number under the blow-up process, see [17], the following equality holds:

(2) $\mu(\mathbb{P}^n; Z, Q) = (d - 1)^n + (-1)^{n-1}(e(D) - h(n - 1, d)) + \mu(\mathcal{X}; \widetilde{Z}, D),$

where $h(n - 1, d)$ is the Euler characteristic of a smooth hypersurface of degree d in $\mathbb{P}^{n-1}$. Finally, (2) is equivalent to the equality in Theorem 1 since the germs $(V, 0)$ and (Z, Q) are topologically equivalent and the generalized Milnor number of D in $\mathbb{P}^{n-1}$ is $\mu(\mathbb{P}^{n-1}; D) = (-1)^{n-1}(e(D) - h(n - 1, d))$.   ∎

Example 4: If $(V, 0)$ is an isolated plane curve singularity we obtain the formula

$$\mu(V, 0) = d(d - 1) + \sum_{\tilde{x} \in \mathrm{Sing}(\widetilde{V})} \mu(\widetilde{V}, \tilde{x}) + 1 - r,$$

where r is the number of different tangent lines of $(V, 0)$ at the origin.

Example 5: Yomdin singularities. Let $f = f_d + f_{d+k} + \cdots$ and $T \subset \mathbb{P}^{n-1}$ be the divisor defined by f_{d+k}. Assume that (∗) $\mathrm{Sing}(D) \cap T = \emptyset$ (which implies that D has only finitely many singular points $\{P_1, \ldots, P_s\}$ which are not in T). If $k = 1$ then $\widetilde{Z}$ is smooth over D and Theorem 1 gives

$$\mu(V, 0) = (d-1)^n + \sum_{i=1}^{s} \mu(D, P_i).$$

If $k > 1$ then singularities of $\widetilde{Z}$ on $\widetilde{Z} \cap E$ are in one-to-one correspondence with singularities of D. If $\widetilde{P}_i$ is the corresponding point to P_i then a local equation of $\widetilde{Z}$ at $\widetilde{P}_i$ is

$$f_d(x_1, \ldots, x_n) + x_0^k\, u(x_0, \ldots, x_n) = 0,$$

where $f_d(x_1, \ldots, x_n) = 0$ is an (affine) equation of D at P_i, $u(0) \neq 0$ and $x_0 = 0$ is an equation for E. Thus

$$\mu(\mathcal{X}; \widetilde{Z}, D) = \sum_{i=1}^{s} \mu(\mathcal{X}; \widetilde{Z}, \widetilde{P}_i) = (k-1) \sum_{i=1}^{s} \mu(D, P_i).$$

By Theorem 1, we obtain the well-known formula for Yomdin singularities, e.g. [14],

$$\mu(V, 0) = (d-1)^n + k \sum_{i=1}^{s} \mu(D, P_i).$$

3. Durfee's conjecture for absolutely isolated surface singularities

Let $(V, 0) \subset (\mathbb{C}^3, 0)$ be an isolated surface singularity given by the zero locus of the germ of a holomorphic function f. Let $\pi \colon M \to V$ be a resolution of the singularity. The **geometric genus** p_g of the singularity is defined by

$$p_g := \dim_{\mathbb{C}} H^1(M, \mathcal{O}_M).$$

In [3], Durfee conjectured that the geometric genus and the Milnor number of the singularity satisfy the inequality $6\, p_g \leq \mu$. This conjecture has been extensively studied in Singularity Theory, nevertheless it is still open. Using Theorem 1 we prove it for any absolutely isolated singularity.

THEOREM 2: *Let $\pi_1 \colon \tilde{V} \to (V, 0)$ be the blow-up at the singular point. Assume that $\tilde{V}$ has only isolated singularities $\{x_1, \ldots, x_s\}$ and Durfee's conjecture is true for each of them $(\tilde{V}, x_i)$. Then Durfee's conjecture is true for $(V, 0)$.*

Proof: Let $\pi \colon M \to \tilde{V}$ be a resolution of the isolated singular points $\{x_1, \ldots, x_s\}$ of $\tilde{V}$. Then the map $\pi_1 \circ \pi \colon M \to V$ is a resolution of the singularity $(V, 0)$.

Following Tomari, [24] Proposition 2.5, the geometric genus p_g of $(V, 0)$ and the geometric genus p_g^i of $(\tilde{V}, x_i)$ satisfy

$$6p_g = \sum_{i=1}^{s} 6\, p_g^i + d(d-1)(d-2),$$

where d is the multiplicity of V at 0. By hypothesis for every singularity $(\tilde{V}, x_i)$ Durfee's conjecture is satisfied, then

$$6p_g \leq \sum_{i=1}^{s} \mu(\tilde{V}, x_i) + d(d-1)(d-2).$$

Let D be the (projectivized) tangent cone of V at 0. The above inequality, the formula in Theorem 1 and the fact that $\mu(\mathbb{P}^2; D) \geq 0$ (Example 2) give the inequality $6\, p_g \leq \mu$ for $(V, 0)$.   ∎

Definition: The surface singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ is **absolutely isolated** if there exists a resolution of $(V, 0)$ obtained only by blowing-ups with a point as center.

THEOREM 3: *Durfee's conjecture is true for any absolutely isolated surface singularity $(V, 0) \subset (\mathbb{C}^3, 0)$.*

Proof: Let $\pi \colon M \to V$ be a resolution obtained only by blowing-ups with a point as center. Let us call **last singularities** those singularities x which appear in this resolution process and such that after blow-up a small neighbourhood U in the strict transform of V with center x the strict transform of U is smooth. After the previous theorem we only need to prove that last singularities verify Durfee's conjecture. But any last singularity is a superisolated singularity, [13], i.e. a Yomdin singularity with $k = 1$. Hence for them we have $6\, p_g = d(d-1)(d-2)$ and $\mu = (d-1)^3 + \mu(\mathbb{P}^2; D)$ where d is the multiplicity of the singularity and D is the (projectivized) tangent cone. So the theorem is proved.   ∎

4. t-singularities

Let us consider a holomorphic function f in three variables with $f(0) = 0$. Let $(V, 0)$ be its zero locus. We need some new notations. Let $f = f_d + f_{d+k} + \cdots$ be the expansion of f as a sum of its homogeneous components. Consider $f_d = h_1^{q_1} \cdots h_s^{q_s}$ to be the decomposition of f_d into irreducible factors. Consider the following divisors on $\mathbb{P}^2$:

(1) For every $i \in \{1, \ldots, s\}$, C_i the projective plane curve defined by h_i,

(2) $D := \mathcal{Z}(f_d) = q_1 C_1 + \cdots + q_s C_s$ the projectivized tangent cone of V at 0,

(3) $D_{red} := C_1 + \cdots + C_s$ and

(4) the divisor T defined by f_{d+k}.

Let d_i be the degree of C_i (then $d = q_1 d_1 + \cdots + q_s d_s$) and let p be the degree of D_{red}; it means that $p = d_1 + \cdots + d_s$. Let Λ be the set of indices i such that q_i is greater than one.

Definition: A surface singularity $(V, 0) \subset (\mathbb{C}^3, 0)$ is called a *t*-**singularity** if there exists a holomorphic function f such that $(V, 0)$ is its zero locus and which verifies the following two conditions:

  (i) The intersection of $\mathrm{Sing}(D_{red})$ and T in $\mathbb{P}^2$ is empty.

  (ii) For every $i \in \Lambda$, the curves C_i and T meet at $d_i (d + k)$ different points.

The t stands for the 'weak' transversality condition (ii) in the above definition. Notice that if Λ is empty then condition (i) is condition $(*)$ for Yomdin singularities. Therefore the following result, which will be proved in this section, is a natural generalization of previous known results for Yomdin singularities.

THEOREM 4: *Every t-singularity $(V, 0)$ is isolated, its Milnor number is equal to*

$$\mu(V, 0) = (d - 1)^3 + k\mu(\mathbb{P}^2; D) + k(d - p)(d + k)$$

and its topological type is determined by the $(d + k)$-jet of f.

Remark that the previous formula for the Milnor number is equivalent to the one appearing in the introduction because $\mu(\mathbb{P}^2; D) = e(D) - 3d + d^2$. We prove this theorem by a sequence of lemmas. Lemma 1 proves that $(V, 0)$ is an isolated singularity. Lemmas 2, 3 and 4 are used for computing its Milnor number. Finally with Lemma 5 we prove that $(V, 0)$ is $(d + k)$-determined.

LEMMA 1: *Every t-singularity $(V, 0)$ has an isolated singular point.*

Proof: It is enough to show that after one-point blow-up all singularities of the strict transform $\widetilde{V}$ are on the exceptional divisor E. Let $\pi: (\mathcal{X}, E) \to (\mathbb{C}^3, 0)$ be the blow-up at 0. If $\mathrm{Sing}(D)$ is empty then $\mathrm{Sing}(\widetilde{V}) \cap E$ is empty too and $(V, 0)$ has an isolated singularity. Otherwise, let $P \in \mathrm{Sing}(D)$. We choose coordinates so that the tangent direction corresponding to P is $(x, y, z) = (0, 0, 1)$. Thus, the local equations of π are $x = x_1 z_1$, $y = y_1 z_1$, $z = z_1$ and an equation of $\widetilde{V}$ in a neighbourhood of the corresponding point $\widetilde{P} = (0, 0, 0)$ is

$$0 = \widetilde{f}(x_1, y_1, z_1) = f_d(x_1, y_1, 1) + z_1^k(f_{d+k}(x_1, y_1, 1) + z_1 g(x_1, y_1, z_1)).$$

Notice that $f_d(x_1, y_1, 1) = 0$ and $z_1 = 0$ are an affine equation of D and a local equation of E respectively.

(a) If $P \in \text{Sing}(D_{red})$ then $f_{d+k}(0, 0, 1) \neq 0$ because $(V, 0)$ is a t-singularity. So $f_d(x_1, y_1, 1)$ is a unit in $\mathbb{C}\{x_1, y_1\}$. Let $u(x_1, y_1)$ be a kth root of this unit. Making the analytic change of coordinates $\bar{x} = x_1$, $\bar{y} = y_1$, $\bar{z} = z_1 u$, the germ of D at $(\bar{x}, \bar{y}) = (0, 0)$ is given by $f_d(\bar{x}, \bar{y}, 1) = 0$, E is given by $\bar{z} = 0$ and $\widetilde{V}$ at $\widetilde{P}$ is defined by $f_d(\bar{x}, \bar{y}, 1) + \bar{z}^k = 0$. Hence singularities of $\widetilde{V}$ in a small neighbourhood of $\widetilde{P}$ lie on E.

(b) For $i \in \Lambda$, let P be a smooth point of C_i which is not in T. There exists an analytic change of coordinates such that $\bar{x} = 0$ is an equation of C_i at P, E is $\bar{z} = 0$ and the equation of the germ of $\widetilde{V}$ at the corresponding point $\widetilde{P}$ is $\bar{x}^{q_i} + \bar{z}^k = 0$. Therefore all singularities of $\widetilde{V}$ around $\widetilde{P}$ are on E.

(c) Let $P \in C_i \cap T$. Since C_i is transverse to T at P we may choose coordinates such that $\bar{x} = 0$ is an equation of C_i at P, the local equation of T is $\bar{y} = 0$, E is $\bar{z} = 0$ and $\widetilde{V}$ at $\widetilde{P}$ is given by $\bar{x}^{q_i} + \bar{z}^k \bar{y} = 0$. The singular locus of $\widetilde{V}$ around $\widetilde{P}$ is again on E. ∎

COMPUTATION OF THE MILNOR NUMBER. We suppose that Λ is not empty, otherwise Example 5 gives the result. Therefore (ii) in the definition of t-singularity implies that T is a reduced divisor in $\mathbb{P}^2$. We start considering the case $k > 1$.

Since $(V, 0)$ has an isolated singularity we may assume that $(V, 0)$ is defined as the zero locus of a polynomial $f = f_d + f_{d+k} + \cdots + f_r$ of degree r in $\mathbb{C}[x, y, z]$. The projective surface $Z \subset \mathbb{P}^3$, defined by the homogeneous polynomial $\bar{f} = w^r g(\frac{x}{w}, \frac{y}{w}, \frac{z}{w})$, has a singular point at $Q = (0 : 0 : 0 : 1)$ which is topologically equivalent to $(V, 0)$. Let $\pi: \mathcal{X} \to \mathbb{P}^3$ be the blow-up at Q, $\widetilde{Z}$ the strict transform of Z and E the exceptional divisor. Then $\text{Sing}(\widetilde{Z}) \cap E$ consists of the curves C_i, $i \in \Lambda$, and the isolated singularities of D. Let $\mathcal{L}$ be the line bundle $\mathcal{O}_{\mathbb{P}^3}(r)$. The homogeneous polynomial $\bar{f}$ gives a global holomorphic section v of $\mathcal{L}$. The surface $\widetilde{Z}$ is the zero locus of the section $s = \pi^*(v) \otimes e^{-d}$ of the line bundle $\pi^*(\mathcal{L}) \otimes \mathcal{E}^{-d}$, where $\mathcal{E}$ denotes the line bundle on $\mathcal{X}$, associated with E and e is a section of $\mathcal{E}$. By Theorem 1, the Milnor number $\mu(V, 0)$ is computed if we know the generalized Milnor number $\mu(\mathcal{X}; \widetilde{Z}, D)$. Next we apply the property (1) of the generalized Milnor number. For this we need a holomorphic global section $s' \in H^0(\mathcal{X}, \pi^*(\mathcal{L}) \otimes \mathcal{E}^{-d})$ and a Whitney stratification $\mathcal{S}$ of $\widetilde{Z}$ such that the zero set of s' is smooth and transverse to $\mathcal{S}$.

For $i \in \Lambda$, we set $M_i := C_i \cap T$ and $S_i := C_i \smallsetminus \{\text{Sing}(D_{red}) \cup M_i\}$. Notice that $\text{Sing}(D_{red})$ and M_i are disjoint sets. Let $\bar{\mathcal{S}}$ be a Whitney stratification of $\widetilde{Z}$ such

that $\operatorname{Reg}(\widetilde{Z}) := \widetilde{Z} \smallsetminus \operatorname{Sing}(\widetilde{Z})$ is a stratum. Let $\mathcal{S}$ be the following partition of $\widetilde{Z}$:

(1) The two-dimensional part $\operatorname{Reg}(\widetilde{Z})$.

(2) The following parts of $\operatorname{Sing}(\widetilde{Z}) \cap D$: one-dimensional parts $\{S_i\}_{i \in \Lambda}$, zero-dimensional parts $\{\tilde{P} \colon P \in \operatorname{Sing}(D_{red})\}$ and $\{\tilde{P} \colon P \in M_i\}_{i \in \Lambda}$.

(3) The same strata of $\bar{\mathcal{S}}$ which are in $\operatorname{Sing}(\widetilde{Z}) \smallsetminus D$.

LEMMA 2: *The previous partition $\mathcal{S}$ is a Whitney stratification of $\widetilde{Z}$.*

Proof: The only problem for $\mathcal{S}$ being a Whitney stratification is over $\operatorname{Reg}(\widetilde{Z})$, S_i and zero-dimensional parts of $\operatorname{Sing}(\widetilde{Z}) \cap D$. Nevertheless we know:

(i) The strata $\operatorname{Reg}(\widetilde{Z})$ and S_i are Whitney regular over every zero-dimensional stratum (see [3], Lemma 1.10).

(ii) To see that the stratum $\operatorname{Reg}(\widetilde{Z})$ is Whitney regular over S_i we use the equivalence between Whitney regular and μ^*−constant, see [22]. Let $a \in S_i$, the equation of $\widetilde{Z}$ at a is $\bar{x}^{q_i} + \bar{z}^k = 0$ and the equations of S_i at a are $\bar{x} = 0$, $\bar{z} = 0$. Thus $\operatorname{Reg}(\widetilde{Z})$ is the family (of germs) of plane curves given by the equation $\bar{x}^{q_i} + \bar{z}^k = 0$, analytically trivial along S_i.   ∎

Remark 1: Notice that $\mathcal{S}$ induces a Whitney stratification $\mathcal{W}$ of Z where $\{Q\}$ and $\operatorname{Reg}(Z) := Z \smallsetminus \operatorname{Sing}(Z)$ are strata. Moreover, if $\operatorname{Reg}(D) := D \smallsetminus \operatorname{Sing}(D)$ denotes the set of smooth points of D and, for $i \in \Lambda$, A_i denotes the set $C_i \smallsetminus \operatorname{Sing}(D_{red})$ then the partition $\mathcal{A}$ which consists of $\operatorname{Reg}(D)$, $\{A_i\}_{i \in \Lambda}$ and zero-dimensional parts $\{\tilde{P} \colon P \in \operatorname{Sing}(D_{red})\}$ is a Whitney stratification of D.

LEMMA 3: *There exists a global section s' of $\pi^*(\mathcal{L}) \otimes \mathcal{E}^{-d}$ such that if Z' is the zero locus of s' then Z' is smooth and transverse to $\mathcal{S}$.*

Proof: Let $\bar{Z} := Z \smallsetminus \{Q\} \subset \mathbb{P}^3$. Let δ be the linear system of projective surfaces of degree r in $\mathbb{P}^3$ such that its multiplicity at Q is greater than or equal to d (Q is the base point of δ). Restricting δ to $\bar{Z}$ and applying the Bertini Theorem (see [6] Corollary 10.9) there exists an open set U_0 of the projective variety δ such that every element Z_1 of U_0 is a smooth surface out of Q and it is transverse to every stratum, different from $\{Q\}$, of the Whitney stratification $\mathcal{W}$ of Z.

Let $\mathcal{C}_d$ be the projective variety of the projective plane curves of degree d. Let U be the nonempty Zariski open set of $\mathcal{C}_d$ consisting of smooth curves G_d which meet D_{red} at $d \cdot p$ different points and such that $G_d \subset \mathbb{P}^2 \smallsetminus \bigcup_{i \in \Lambda} M_i$.

Let U_1 be the open subset of the projective variety δ which consists on those surfaces such that their projectivized tangent cone at Q belongs to U.

Every surface $Z_1 \in U_0 \cap U_1$ has only one isolated singular point at Q. Let s_1 be a global holomorphic section of $\mathcal{L}$ that defines Z_1. Then $s' := \pi^*(s_1) \otimes e^{-d}$

defines a global holomorphic section of $\pi^*(\mathcal{L}) \otimes \mathcal{E}^{-d}$ and its zero locus $Z' \subset \mathcal{X}$ holds the following properties:

(a) Z' is a smooth surface in $\mathcal{X}$ because Z_1 has only one singular point at Q and its projectivized tangent cone at this point is a smooth curve, see [22].

(b) The surface Z' is transverse to $\mathcal{S}$. We only need look at the local equations of $\widetilde{Z}$ and Z' at their intersection points.   ∎

Lemmas 2 and 3 and the equality (1) show that

$$(3) \quad \mu(\mathcal{X}; \widetilde{Z}, D) = \sum_{i \in \Lambda} \left(\mu_{S_i}(\widetilde{Z}) e(S_i \smallsetminus Z') + \sum_{P \in M_i} \mu_{\widetilde{P}}(\widetilde{Z}) \right) + \sum_{P \in \mathrm{Sing}(D_{red})} \mu_{\widetilde{P}}(\widetilde{Z}).$$

Remark 2: Following the notation in Lemma 3, Remark 1 and (1) show that, for every smooth curve G_d in U, the generalized Milnor number of D is equal to

$$(4) \qquad \mu(\mathbb{P}^2; D) = \sum_{i \in \Lambda} \mu_{A_i}(D) e(A_i \smallsetminus G_d) + \sum_{P \in \mathrm{Sing}(D_{red})} \mu(D, P).$$

LEMMA 4: *Let S_i and A_i be one-dimensional strata of $\mathcal{S}$, respectively $\mathcal{A}$. Then*

$$\mu_{S_i}(\widetilde{Z}) = -(q_i - 1)(k - 1), \quad \mu_{A_i}(D) = -(q_i - 1);$$
$$e(S_i \smallsetminus Z') = e(A_i \smallsetminus G_d) - d_i(d + k).$$

Proof: For every $a \in S_i$, we choose coordinates $(\bar{x}, \bar{y}, \bar{z})$ around a such that the equation of $\widetilde{Z}$ at a is $\bar{x}^{q_i} + \bar{z}^k = 0$ and S_i is given by $\bar{x} = 0$, $\bar{z} = 0$. By Example 1, $\mu_{A_i}(D) = -(q_i - 1)$ and $\mu_{S_i}(\widetilde{Z}) = \mu(\widetilde{Z}, a) = -(q_i - 1)(k - 1)$. Since S_i and Z' meet at $d_i d$ different points, $e(S_i \smallsetminus Z') = e(S_i) - d_i d$. Finally for $i \in \Lambda$, the cardinality of M_i is $d_i(d + k)$ and we find that

$$e(S_i \smallsetminus Z') = e(S_i) - d_i d = e(A_i) - d_i(d + k) - d_i d = e(A_i \smallsetminus G_d) - d_i(d + k). \quad ∎$$

We look at the local equations of $\widetilde{Z}$ at the zero-dimensional strata; see Lemma 1.

(i) If $P \in \mathrm{Sing}(D_{red})$ then $\widetilde{Z}$ at $\tilde{P}$ is given by $h(\bar{x}, \bar{y}) + \bar{z}^k = 0$, where $h = 0$ is an equation of the germ of D at P. Example 1 shows that

$$(5) \qquad \mu_{\widetilde{P}}(\widetilde{Z}) = \mu(\widetilde{Z}, \widetilde{P}) = (k - 1)\mu(D, 0).$$

(ii) If $P \in M_i$ then $\widetilde{Z}$ at $\widetilde{P}$ is defined by $\bar{x}^{q_i} + \bar{z}^k \bar{y} = 0$, where $x = 0$ is an equation of C_i at P. By Example 1, $\mu_{\widetilde{P}}(\widetilde{Z}) = (q_i - 1)$ and since the cardinality of M_i is $d_i(d + k)$ then

$$(6) \qquad \sum_{P \in M_i} (q_i - 1) = (q_i - 1)(d + k)d_i.$$

Lemma 4, identities (5) and (6) show that (3) is equivalent to

$$\mu(\mathcal{X}; \widetilde{Z}, D) = \sum_{i \in \Lambda} -(q_i - 1)(k-1)(e(A_i \smallsetminus G_d) - d_i(d+k))$$

$$+ (d+k) \sum_{i \in \Lambda} (q_i - 1)d_i + (k-1) \sum_{P \in \mathrm{Sing}(D_{red})} \mu(D, P)$$

$$= (k-1) \sum_{i \in \Lambda} -(q_i - 1)e(A_i \smallsetminus G_d) + \sum_{P \in \mathrm{Sing}(D_{red})} \mu(D, P)$$

$$+ k(d+k) \sum_{i=1}^{r} (q_i - 1)d_i.$$

The last equality and (4) give the identity

(7) $$\mu(\mathcal{X}; \tilde{Z}, D) = (k-1)\mu(\mathbb{P}^2; D) + k(d+k)(d-p).$$

Substituting (7) into the identity in Theorem 1 we find that

$$\mu(V, 0) = (d-1)^3 + k\mu(\mathbb{P}^2; D) + k(d+k)(d-p).$$

In case $k = 1$, we consider a Whitney stratification and a global holomorphic section as above (both choices do not depend on k). Singularities of $\widetilde{Z} \cap E$ are corresponding to points of the strata M_i, i.e. points in $C_i \cap T$. The equation of $\widetilde{Z}$ at such a point $\widetilde{P} = (0, 0, 0) \in M_i$ is $x^{q_i} + zy = 0$. Thus $\mu(\widetilde{Z}, \widetilde{P}) = q_i - 1$. Since the cardinality of M_i is $d_i(d+1)$ then (3) can be written as follows:

$$\mu(\mathcal{X}; \widetilde{Z}) = \sum_{i \in \Lambda} \sum_{P \in M_i} \mu_{\tilde{P}}(\widetilde{Z}) = \sum_{q_i > 1} (q_i - 1)d_i(d+1) = (d+1)(d-p).$$

This equality is substituted into the identity in Theorem 1 obtaining the Milnor number of $(V, 0)$. So the computation of $\mu(V, 0)$ is done.

TOPOLOGICAL DETERMINACY ORDER FOR t-SINGULARITIES. It is clear from the formula for the Milnor number just proved that $\mu(V, 0)$ only depends on D and k. Next we show that the Milnor number of a generic plane section of $(V, 0)$ only depends on D and k too.

LEMMA 5: *Let $f = f_d + f_{d+k} + \cdots \in \mathbb{C}\{x, y, z\}$ be the germ of a complex analytic function. If its zero locus $(V, 0)$ has an isolated singular point and the divisors D, defined by f_d, and T, defined by f_{d+k}, have no common components then the Milnor number of a generic plane section of $(V, 0)$ is*

$$\mu^{(2)}(V, 0) = (d-1)^2 + k \cdot (d-p),$$

where p is the degree of the (reduced) plane curve D_{red}.

Proof: Let H be a generic plane in $(\mathbb{C}^3, 0)$, or a line in $\mathbb{P}^2$. The divisor $D_{red} + T_{red}$ in $\mathbb{P}^2$ is reduced because both curves have no common components. Assume that H has equation $z = 0$. The germ $(V \cap H, 0)$ is given by $f(x, y, 0) = f_d(x, y, 0) + f_{d+k}(x, y, 0) + \cdots = 0$. Since H is transverse to $D_{red} + T_{red}$ every root of $f_d(x, y, 0)$ is not a root of $f_{d+k}(x, y, 0)$. Applying Example 5 for plane curves the proof of this lemma is completed (note that singularities of $f_d(x, y, 0) = 0$ are its multiple roots and their Milnor numbers are their multiplicities minus one). ∎

End of the proof of Theorem 4: Let $(V, 0)$ be a t-singularity defined by $f = f_d + f_{d+k} + g$, where g has order greater than $d+k$. Consider the family of surfaces $\{(V_s, 0)\}_{s \in [0,1]}$, where $(V_s, 0)$ is defined by the zero locus of $f_s = f_d + f_{d+k} + sg$. By the previous computation of the Milnor number and Lemma 5, the sequence of Milnor numbers $\mu^*(s) = (\mu(V_s, 0), \mu^{(2)}(V_s, 0), d-1)$ does not depend on s. Hence this family is topologically trivial, [22], and $(V, 0)$ is topologically equivalent to $(V_0, 0)$ which is defined by the zero locus of $f_d + f_{d+k}$. Therefore Theorem 4 is proved. ∎

5. Singularities $f_d + f_{d+k}$

The topological determinacy order of any t-singularity is $(d + k)$. This section is devoted to understanding germs of surface in $(\mathbb{C}^3, 0)$ defined by polynomials with only two homogeneous components. Let $f = f_d + f_{d+k} \in \mathbb{C}\{x, y, z\}$ be such a polynomial and $(V, 0) \subset (\mathbb{C}^3, 0)$ its zero locus.

THEOREM 5: *$(V, 0)$ has an isolated singularity if and only if $\mathrm{Sing}(D) \cap \mathrm{Sing}(T)$ is empty. If it is the case, its Milnor number is given by the formula*

$$\mu(V, 0) = (d-1)^3 + k\Big(\mu(D_{red}) + (d-p)(d+p-3) + \sum_{P \in \mathrm{Sing}(D) \cap T} ((D, T)_P - 1)\Big),$$

or equivalently,

$$\mu(V, 0) = (d-1)^3 + k\Big(e(D) + d^2 - 3d + \sum_{P \in \mathrm{Sing}(D) \cap T} ((D, T)_P - 1)\Big),$$

where $(D, T)_P$ denotes the intersection multiplicity of both curves at P.

Notice that the formula in Theorem 4 can be obtained from the previous one.

In order to prove the theorem we prove firstly that $(V, 0)$ has an isolated singularity and after that we compute its Milnor number.

LEMMA 6: $(V, 0) \subset (\mathbb{C}^3, 0)$ *has an isolated singularity if and only if* $\mathrm{Sing}(D) \cap \mathrm{Sing}(T)$ *is empty.*

Proof: The "only if" part is trivial. For the "if" part let $\pi \colon (\mathcal{X}, E) \to (\mathbb{C}^3, 0)$ be the blow-up at the origin. Again if $\mathrm{Sing}(D)$ is empty then $(V, 0)$ has an isolated singular point. Let P be a singularity of D. If P is not in T, following the proof of Lemma 1, then singularities of $\widetilde{V}$ in a neighbourhood of $\bar{P}$ are on E. Hence let P be a singular point for D and a smooth point for T. An equation of $\widetilde{V}$ in a neighbourhood of $\bar{P}$ is

$$f_d(x_1, y_1, 1) + z_1^k \cdot f_{d+k}(x_1, y_1, 1) = 0.$$

Since T is smooth at P, there exists an analytic change of coordinates such that $\widetilde{P}$ has coordinates $(0, 0, 0)$, an equation for $\widetilde{V}$ at $\widetilde{P}$ is

$$(8) \qquad \bar{y}_1^s + \bar{x}_1 \cdot g(\bar{x}_1, \bar{y}_1) + z_1^k \cdot \bar{x}_1 = 0,$$

and E is again given by $z_1 = 0$. Hence the singular locus of $\widetilde{V}$ around $\widetilde{P}$ lies on E. ∎

COMPUTATION OF THE MILNOR NUMBER. If D is a reduced divisor in $\mathbb{P}^2$ then all singularities of D are isolated and $\mu(\mathbb{P}^2; D)$ is the sum of the Milnor numbers of D at its singular points. Thus we need to check that

$$\mu(V, 0) = (d - 1)^3 + k \sum_{P \in \mathrm{Sing}(D)} \mu(D, P) + k \sum_{P \in \mathrm{Sing}(D) \cap T} ((D, T)_P - 1).$$

After the origin blow-up all singularities of $\widetilde{Z}$ are isolated. Moreover, Example 2 shows that for every $P \in \mathrm{Sing}(D) \cap T$ we find that $\mu(\mathcal{X}; \widetilde{Z}, \widetilde{P}) = (k-1)\mu(D, P) + k(s - 1)$, where s is the intersection multiplicity of D and T at P. We finish the proof using Theorem 1.

In case D is a nonreduced divisor, to prove Theorem 5 we use the same strategy as in Theorem 4. Consider the strict transform $\tilde{Z}$ after one blow-up; look at an equation of the germ of $\tilde{Z}$ at its singular points which are on the exceptional divisor E. Then the only different local situation from that in Theorem 4 is at those points $P \in \mathrm{Sing}(D_{red}) \cap T$. In (8), a local equation for $\tilde{Z}$ at its corresponding point $\widetilde{P}$ is given and Example 2 gives its topological Milnor number. After this small change, all the proof is similar to the one in Theorem 4. ∎

Examples: The statement about the topological determinacy order is not true in this more general set-up; see the next examples of surfaces defined by two homogeneous terms.

1. Let $f = y^2z + x^2z + x^3 + z^3x + y^4 + x^4$ and $(V, 0)$ the singularity defined by f. Theorem 5 gives $\mu(V, 0) = 10$. Let $g = f + \frac{1}{4}z^5$. By Theorem 1 the Milnor number of the singularity defined by g is 11; it means that $(V, 0)$ is not $(d+k)$-determined.

2. Let f be the polynomial $y^2(y + z) + x^3 + z^3x + y^4$ and let $(V, 0)$ be its zero locus. The supremum of the quotient of its polar invariants is $\frac{7}{2}$ and that means that $(V, 0)$ is $(d + k)$-determined, see [23].

6. Some projective surfaces

We have just remarked in Theorem 1 that if $(V, 0)$ is an isolated hypersurface singularity then the generalized Milnor number of its projectivized tangent cone at the origin is related to its Milnor number. This tangent cone is a positive divisor in $\mathbb{P}^{n-1}$. It seems interesting to apply our results in Theorem 5 to study projective surfaces in $\mathbb{P}^3$ because such a surface may appear as a tangent cone of a 3-dimensional singularity. The results found in this section deal with the classification problem but from a global point of view. Therefore we consider a projective surface $Z \subset \mathbb{P}^3$ defined by

$$Z := \{(x : y : z : w) \in \mathbb{P}^3 : f_d(x, y, z)\, w^k + f_{d+k}(x, y, z) = 0\}.$$

We give necessary and sufficient conditions for Z to have isolated singularities. Furthermore, we obtain a total description of its singularities in terms of their Milnor numbers. The case $k = 1$ may be treated from Theorem 5 and from some results of Soares and Giblin, [21]. So from now on we assume k greater than one.

THEOREM 6: *Let $Z \subset \mathbb{P}^3$ be a projective surface defined by $f_d w^k + f_{d+k} = 0$. The surface Z has only isolated singularities if and only if T is a reduced plane curve and $\mathrm{Sing}(D) \cap \mathrm{Sing}(T)$ is empty. If this is the case the singularities of Z are:*

(1) *The point $e = (0 : 0 : 0 : 1)$ whose Milnor number is given in Theorem 5.*

(2) *For each $P \in \mathrm{Sing}(T) \smallsetminus D$, the point $\bar{P} := (P : 0)$ is a singular point of Z and $\mu(Z, \bar{P}) = (k - 1)\mu(T, P)$.*

(3) *For each $P \in D \cap \mathrm{Sing}(T)$, $\bar{P} := (P : 0)$ is a singularity of Z and $\mu(Z, \bar{P}) = (k - 1)\mu(T, P) + k((D, T)_P - 1)$.*

(4) *For each $P \in D \cap T$ which is a smooth point for both curves and satisfies $(D, T)_P > 1$, every point $\bar{P}^i := (P : w_i)$ is a singular point of Z, where for*

$i \in \{1, \ldots, k\}$, w_i *verifies the equality* $w_i^k \operatorname{grad}(f_d)(P) + \operatorname{grad}(f_{d+k})(P) = 0$. *Moreover,* $\mu(Z, \bar{P}^i) = (D, T)_P - 1$.

Therefore the generalized Milnor number of Z, i.e. the sum of all local Milnor numbers, is equal to

$$\mu(\mathbb{P}^3; Z) = (d-1)^3 + k\mu(\mathbb{P}^2; D) + (k-1)\mu(\mathbb{P}^2; T) + k\,d\,(d+k) - k\,\#(D \cap T),$$

where $\#$ denotes the cardinality of the intersection set $D \cap T$.

LEMMA 7: *The surface Z has only isolated singularities if and only if the plane curve T is reduced and* $\operatorname{Sing}(D) \cap \operatorname{Sing}(T)$ *is empty.*

Proof: The singular locus of Z is defined by the equations

$$w^k \operatorname{grad}(f_d) + \operatorname{grad}(f_{d+k}) = 0,$$
$$k w^{k-1} f_d = 0.$$

The curve T must be reduced, otherwise any multiple component L of T gives a singular curve of Z. By the other way, if there exists $P = (a_0 : a_1 : a_2)$ such that $P \in \operatorname{Sing}(D) \cap \operatorname{Sing}(T)$, then the point $P \in \mathbb{P}^2$ defines a line L_P in $\mathbb{P}^3$ (L_P is the projective closure in $\mathbb{P}^3$ of the affine line $\{(a_0 : a_1 : a_2 : w) \colon w \in \mathbb{C}\}$) which is singular for Z.

For the "only if" part we know, by Theorem 2, that $e = (0 : 0 : 0 : 1)$ is a singularity of Z. Let a be another singular point of Z. We may write $a = (P : w)$, where P is a point in $\mathbb{P}^2$. For the last equation of the singular locus to vanish, we have only two possibilities:

(i) The coordinate w is zero. Then $\operatorname{grad}(f_{d+k})(P)$ is zero and this implies that P is a singular point of T. Since T is a reduced divisor it has only finitely many singular points.

(ii) $P \in D$. Looking at the equation that defines Z, we find that $P \in T$. Since T is reduced, $\operatorname{Sing}(D) \cap \operatorname{Sing}(T)$ is empty and the degree of T is greater than the degree of D, curves D_{red} and T have no common component. The Bezout Theorem shows that there is only a finite number of points in $D \cap T$ (P is one of them). Next we prove that for each of them there exists finitely many complex numbers w verifying the equations of the singular locus. From the first one P is a smooth point of D.

(1) If $P \in \operatorname{Sing}(T)$ then the w-coordinate of a is zero and we are again in (i).

(2) If P is a (smooth) point for both curves then the w-coordinate of a is different from zero. Let $\lambda \in \mathbb{C}^*$ be a non-zero complex number such that

$$(9) \qquad \lambda \cdot \left(\frac{\partial f_d}{\partial x}, \frac{\partial f_d}{\partial y}, \frac{\partial f_d}{\partial z} \right)(P) = \left(\frac{\partial f_{d+k}}{\partial x}, \frac{\partial f_{d+k}}{\partial y}, \frac{\partial f_{d+k}}{\partial z} \right)(P).$$

If λ exists, then it is unique because it gives the proportionality between two non-zero vectors $\mathrm{grad}(f_d(P))$ and $\mathrm{grad}(f_{d+k}(P))$. Furthermore, if $\lambda \in \mathbb{C}^*$ exists then there are exactly k different values for w such that $w^k = \lambda$ and $a : (P : w) \in \mathrm{Sing}(Z)$. ∎

LEMMA 8: *For every $P \in D \cap T$ such that P is a smooth point for both curves, there exists $\lambda \in \mathbb{C}^*$ satisfying (9) if and only if D and T have the same tangent line at P.*

Proof: After a suitable change of homogeneous coordinates we may assume that $P = (0 : 0 : 1)$, and $a = (0 : 0 : 1 : w_0)$, $w_0 \neq 0$. Since P is a smooth point for both curves we deduce that $f_d(x, y, 1) = \alpha x + \beta y + h(x, y)$ and $f_{d+k}(x, y, 1) = \gamma x + \delta y + g(x, y)$ where h and g have order at the origin greater than one. Hence the identity (9) gives the equivalence in Lemma 8. ∎

Lemmas 7 and 8 show that all singularities of Z are:
(1) The point $e = (0 : 0 : 0 : 1)$,
(2) $(P : 0)$, where $P \in \mathrm{Sing}(T)$,
(3) for each Q which is a smooth point for D and T and such that $(D, T)_Q > 1$, Z has k singularities, each of which is $(Q : w_0)$ where w_0 has been obtained as in Lemma 7.

LEMMA 9: *Let $P \in \mathrm{Sing}(T) \smallsetminus D$; the Milnor number of Z at $\bar{P} := (P : 0)$ is equal to $\mu(Z, \bar{P}) = (k - 1)\mu(T, P)$.*

Proof: We choose projective coordinates in $\mathbb{P}^2$ such that $P = (0 : 0 : 1)$. An affine equation of Z at $\bar{P}$ is $f_d(x, y, 1)w^k + f_{d+k}(x, y, 1) = 0$. Example 1 gives the formula for the Milnor number. ∎

LEMMA 10: *Let $P \in D \cap \mathrm{Sing}(T)$; the Milnor number of Z at $\bar{P} := (P : 0)$ is*

$$\mu(Z, \bar{P}) = (k - 1)\mu(T, P) + k((D, T)_P - 1).$$

Proof: The proof follows from Example 2. ∎

LEMMA 11: *Let $P \in D \cap T$ be a smooth point for both curves such that the intersection multiplicity $(D, T)_P > 1$, the Milnor number of Z at $\bar{P}^0 := (P : w_0)$ is equal to $\mu(Z, \bar{P}^0) = (D, T)_P - 1$, where w_0 verifies the equality $w_0^k \, \mathrm{grad}(f_d)(P) + \mathrm{grad}(f_{d+k})(P) = 0$.*

Proof: We may assume that $P = (0 : 0 : 1)$; then $\bar{P}^0 := (0 : 0 : 1 : w_0)$, $w_0^k \neq 0$ and an affine equation of Z in the open set $\mathbb{P}^3 \smallsetminus \{z = 0\}$ is given by $f_d(x, y, 1)w^k + f_{d+k}(x, y, 1) = 0$. In this affine subset $\bar{P}^0$ has affine coordinates

$(0, 0, w_0)$. For computing the Milnor number of Z at $\bar{P}^0$ we work in the local ring $\mathbb{C}\{x, y, w - w_0\}$. Since D is smooth at P there exists an analytic change of coordinates $h_1 \colon (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ such that $f_d(h_1(\bar{x}, \bar{y}), 1) = \bar{x}$ and $\rho(\bar{x}, \bar{y}) :=$ $f_{d+k}(h_1(\bar{x}, \bar{y}), 1)$ satisfies

$$\rho(\bar{x}, \bar{y}) = v(\bar{x}, \bar{y}) \cdot (\bar{y}^s + \bar{x}b_1(\bar{x})\bar{y}^{s-1} + \cdots + \bar{x}b_{s-1}(\bar{x})\bar{y} + \bar{x}b_s(\bar{x})),$$

where $v(0, 0) \neq 0$ and s is the intersection multiplicity of both curves at the origin. Let $h \colon (\mathbb{C}^3, (0, 0, w_0)) \to (\mathbb{C}^3, (0, 0, w_0))$ be the change of coordinates $h(\bar{x}, \bar{y}, \bar{w}) = (h_1(\bar{x}, \bar{y}), \bar{w})$. Let p be the function $f \circ h = \bar{x}w^k + \rho(\bar{x}, \bar{y})$. Since $\bar{P}^0$ is singular for Z we deduce that $0 = p_{\bar{x}}(0, 0, w_0) = v(0, 0)b_s(0) + w_0^k$ and then $v(0, 0)b_s(0) = -w_0^k$. Let us expand p as a power series around $(0, 0, w_0)$:

$$p(\bar{x}, \bar{y}, \bar{w}) = \rho(\bar{x}, \bar{y}) + \bar{x}\left(w_0^k + \sum_{j=1}^{k} \frac{k(k-1)\cdots(k-j+1) \cdot w_0^{k-j}}{i!}(\bar{w} - w_0)^j\right).$$

Making the change of coordinates $\bar{x}_1 = \bar{x}, \bar{y}_1 = \bar{y}, \bar{w}_1 = \bar{w} - w_0$, the lifting $\bar{p}$ of p in the new ring $\mathbb{C}\{\bar{x}_1, \bar{y}_1, \bar{w}_1\}$ is

$$\bar{p} = \rho(\bar{x}_1, \bar{y}_1) + \bar{x}_1\left(w_0^k + \sum_{j=1}^{k} \frac{k(k-1)\cdots(k-j+1) \cdot w_0^{k-j}}{j!}\bar{w}_1^j\right)$$

$$= \rho(\bar{x}_1, \bar{y}_1) + \bar{x}_1\left(w_0^k + \bar{w}_1\sum_{j=1}^{k} \frac{k(k-1)\cdots(k-j+1) \cdot w_0^{k-j}}{j!}\bar{w}_1^{j-1}\right).$$

Since w_0 is different from zero, the series

$$\sum_{j=1}^{k} \frac{k(k-1)\cdots(k-j+1) \cdot w_0^{k-j}}{j!}\bar{w}_1^{j-1}$$

is a unit in $\mathbb{C}\{\bar{w}_1\}$. Hence the change of coordinates

$$\bar{x}_2 = \bar{x}_1,$$
$$\bar{y}_2 = \bar{y}_1,$$
$$\bar{w}_2 = \bar{w}_1\left(\sum_{j=1}^{k} \frac{k(k-1)\cdots(k-j+1) \cdot w_0^{k-j}}{j!}\bar{w}_1^{j-1}\right)$$

gives the lifting $\bar{p}_1$ of $\bar{p}$ in the ring $\mathbb{C}\{\bar{x}_2, \bar{y}_2, \bar{w}_2\}$,

$$\bar{p}_1(\bar{x}_2, \bar{y}_2, \bar{w}_2) = \rho(\bar{x}_2, \bar{y}_2) + \bar{x}_2(w_0^k + \bar{w}_2).$$

Moreover, $\bar{p}_1(\bar{x}_2, \bar{y}_2, \bar{w}_2)$ can be written as follows:

$$\bar{x}_2 \left(w_0^k + \bar{w}_2 + v(\bar{x}_2, \bar{y}_2) \left(b_1(\bar{x}_2)\bar{y}_2^{s-1} + \cdots + b_{s-1}(\bar{x}_2)\bar{y}_2 + b_s(\bar{x}_2) \right) \right) + v(\bar{x}_2, \bar{y}_2)\bar{y}_2^s.$$

The following equations define the last change of analytic coordinates:

$$\bar{x}_3 = \bar{x}_2,$$
$$\bar{y}_3 = \bar{y}_2,$$
$$\bar{w}_3 = \bar{w}_2 + v(\bar{x}_2, \bar{y}_2) \left(b_1(\bar{x}_2)\bar{y}_2^{s-1} + \cdots + b_{s-1}(\bar{x}_2)\bar{y}_2 + b_s(\bar{x}_2) \right) + w_0^k,$$

because $v(0,0)b_s(0) + w_0^k = 0$. After this change of coordinates an equation for the germ of Z at $\bar{P}^0 = (0,0,0)$ is

$$x_3 w_3 + v(x_3, y_3)y_3^s = 0,$$

where $v(0,0) \neq 0$. So the Milnor number of Z at $\bar{P}^0$ is $(s-1)$. ∎

Therefore Theorem 5 and Lemmas 7 to 11 give the proof of Theorem 6.

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